Department of Mathematics
University of Pavia, Italy

Economic effects of trading uncertainties

Giuseppe Toscani

Joint work with E. Ballante, C. Bardelli, S. Figini, M. Zanella

Pavia, April 4, 2020

Outline

1. Kinetic modeling
2. Experiments

Outline

1. Kinetic modeling

2. Experiments

The model of wealth distribution

- We consider a linear kinetic equation in which the change of wealth $v \in, \mathbb{R}_+$ is consequent to interactions with a fixed environment.
- The variation of wealth in a single (microscopic) interaction is given by

$$v^* = v + \lambda(x)(w - v) + \eta v.$$

- For $0 \le x \le 1$, $0 < \lambda_1 < \lambda(x) < \lambda_2 < 1$ is the uncertain transaction rate parameter. In this case the saving propensity equals $1 - \lambda(x)$.
- The quantity η is a random variables with mean zero and variance σ. It models risky investments that each agent performs. To guarantee that the post-trade wealths are non-negative, it is assumed

$$\eta > 1 - \lambda_1.$$

The kinetic equation

- Let $f = f(v,t)$ denote the density of agents with wealth v at time $t > 0$.
- The density $f(v,t)$ satisfies, for all smooth functions $\varphi(v)$ (the observable quantities)

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(t)\varphi(v)\,dv = \left\langle \frac{1}{\tau}\int_{\mathbb{R}_+\times\mathbb{R}_+} (\varphi(v^*) - \varphi(v))\, f(v,t)\mathcal{E}(w)\,dv\,dz \right\rangle.$$

- The function $\mathcal{E}(w)$, $z \in \mathbb{R}_+$ denotes the distribution of the wealth in the environment and τ is the frequency parameter.
- $\langle\cdot\rangle$ represents mathematical expectation. Here expectation takes into account the presence of the random parameter η.
- The kinetic equations describe the evolution of the density, and allow us to study the long-time behavior of the system of agents.

Fokker–Planck limit

- Given a small parameter ϵ, consider the scaling

$$\lambda(x) \to \epsilon\lambda(x), \quad \eta, \to \epsilon^{1/2}\eta, \quad \tau \to \epsilon\tau.$$

- Letting $\epsilon \to 0$ the solution $f_\epsilon(v,t)$ of the ϵ-dependent kinetic equation converges to the solution $h(v,t)$ of the Fokker–Planck equation [Cordier, Pareschi, G.T., J. Stat. Phys. (2005)]

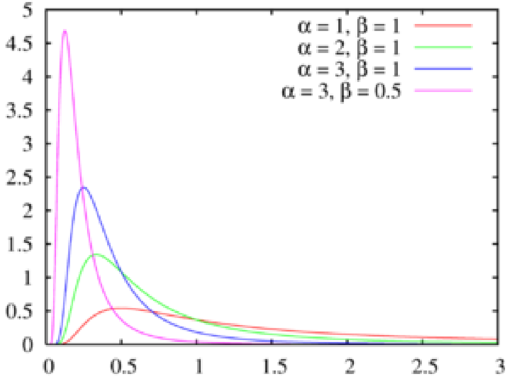
$$\frac{\partial h}{\partial \tau} = \frac{\sigma}{2}\frac{\partial^2}{\partial v^2}\left(v^2 h\right) + \lambda(x)\frac{\partial}{\partial v}\left((v-1)h\right).$$

- The steady state is the inverse Gamma distribution

$$h_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)}\frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}}.$$

- The function $\mu(x) = 2\lambda(x)/\sigma$ in an uncertain parameter. The steady state has polynomial tails.

Gamma Inversa



The saving parameter

- The change of wealth $v \in, \mathbb{R}_+$ depends on two random quantities. The saving $1 - \lambda(x)$, and the risk variable η.
- We assume that the risk variable η is universal.
- On the contrary, the personal saving propensity of agents can vary for various reasons, linked for example to stressed situations, which modify the opinion of agents with respect to trading.
- Important to couple the trading with the opinion.
- We consider the time scale of opinion formation of higher order with respect to wealth trading.

Opinion formation on saving

- To model opinion formation, we assume the classical interaction given by [G.T., Commun. Math. Sci., (2006)]

$$x^* = x + \mu(z - x) + \eta\sqrt{(1 - x^2)}, \quad x \in [-1, 1]$$

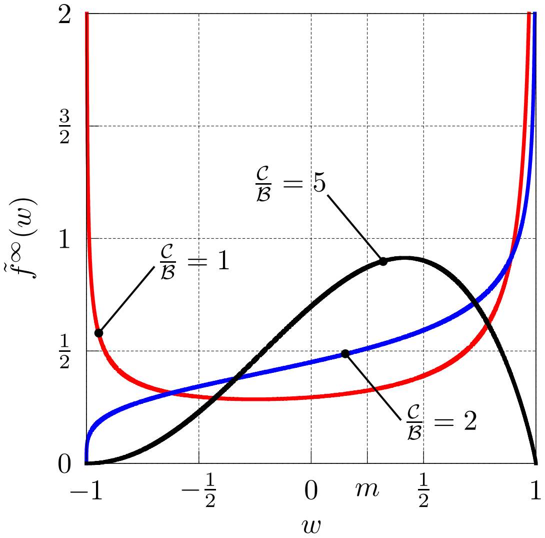
- In this case η models the self-thinking, and $-1 < \mu < 1$ the rate of compromise with the opinion of the environment z.
- in the quasi-invariant opinion limit the opinion density $g(x, t)$ satisfies the Fokker–Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}\left((1 - x^2)g\right) + \mu\frac{\partial}{\partial x}\left((x - m)g\right).$$

- If $\gamma = 2\mu/\sigma$, and $-1 < m < 1$ is the mean opinion of the environment, the steady state is the Beta distribution (with support in $[-1, 1]$)

$$g_\infty(x) = c_{m,\gamma}(1 + x)^{(1+m)/\gamma - 1}\,(1 - x)^{(1-m)/\gamma - 1}.$$

Beta



2
3/2
1
1/2
0
$\tilde{f}^\infty(w)$
$\frac{\mathcal{C}}{\mathcal{B}} = 5$
$\frac{\mathcal{C}}{\mathcal{B}} = 1$
$\frac{\mathcal{C}}{\mathcal{B}} = 2$
−1
−1/2
0
m
1/2
1
w

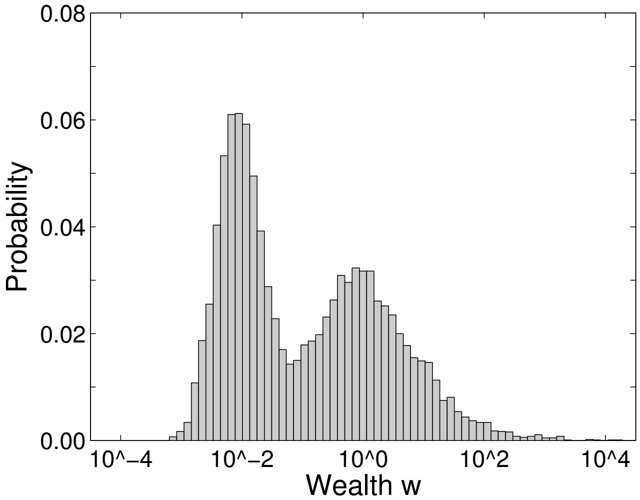
Extremal cases

- The Beta distribution allows to consider a variety of situations.
- If $\gamma \to 0$, the limit density is given by a one-valued random variable (a Dirac delta in $x = m$). This corresponds to the standard case with a fixed constant value $\tilde{\lambda}$.
- All agents in the system have the same saving propensity. The stationary wealth distribution is a standard inverse Gamma.
- If we set $\gamma \to \infty$, the limit density is given by a two-valued random variable X, such that

$$P(X = -1) = p = \frac{1}{2}(1 - m), \quad P(X = +1) = q = \frac{1}{2}(1 + m).$$

- Agents in the system split into two groups, each one with constant saving propensity. The stationary wealth distribution is a mixture of two inverse Gamma with parameters λ_1 and λ_2 [A.K. Gupta (2006)].

Bimodal distribution



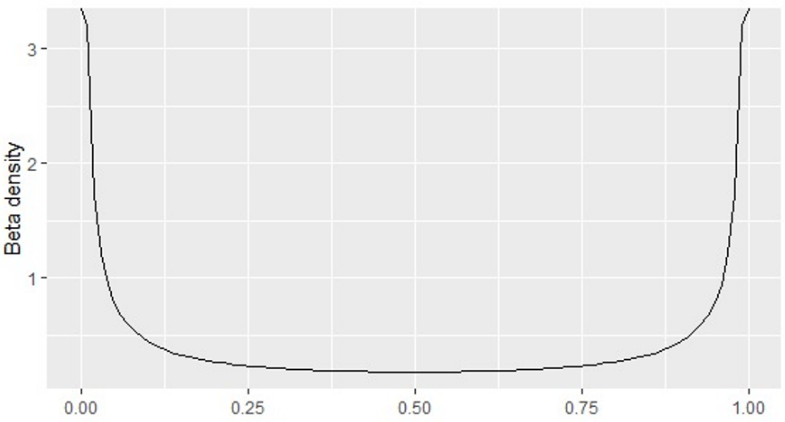
When bimodal distribution?

- If Y is a Beta distribution supported in $[-1,1]$, $X = (Y+1)/2$ is a Beta distribution supported in $[0,1]$.
- Let us denote $B(\alpha,\beta)$ the Beta distribution

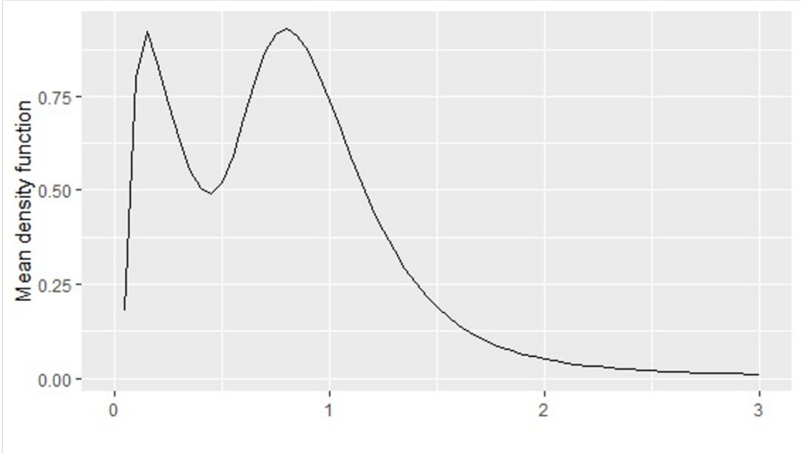
$$B(\alpha,\beta) = c_{\alpha,\beta}x^{\alpha-1}(1-x)^{\beta-1}, \quad \alpha,\beta \geq 0.$$

- Numerical experiments show that the bimodal shape remains if $\alpha,\beta \leq 0.6$, provided the support of $\lambda(x)$ in the steady state of wealth density is sufficiently large.
- This includes non-extremal situations with a bimodal shape.
- The variations of Pareto index are under study.

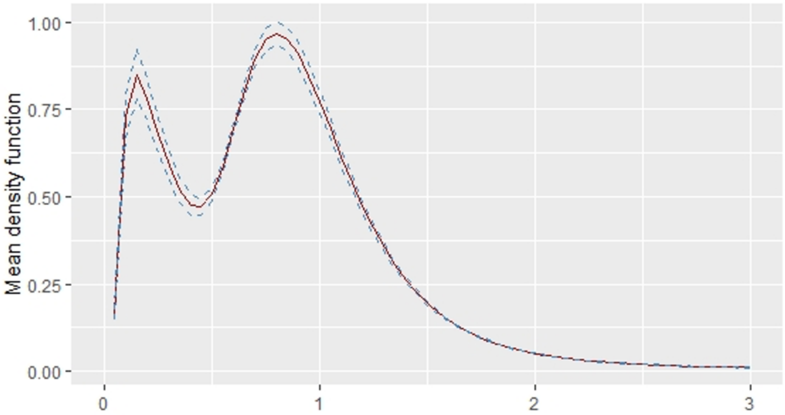
Beta B(0.1,0.1)



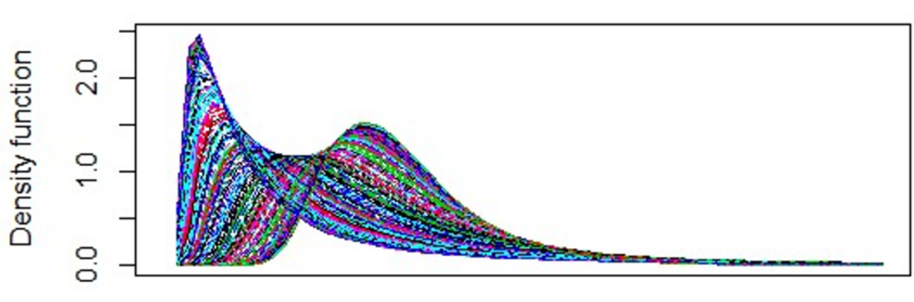
Bimodal distribution



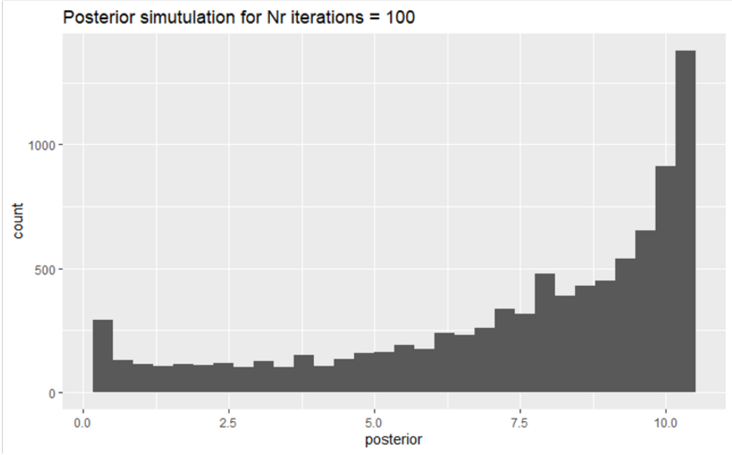
Confidence interval of 95/100



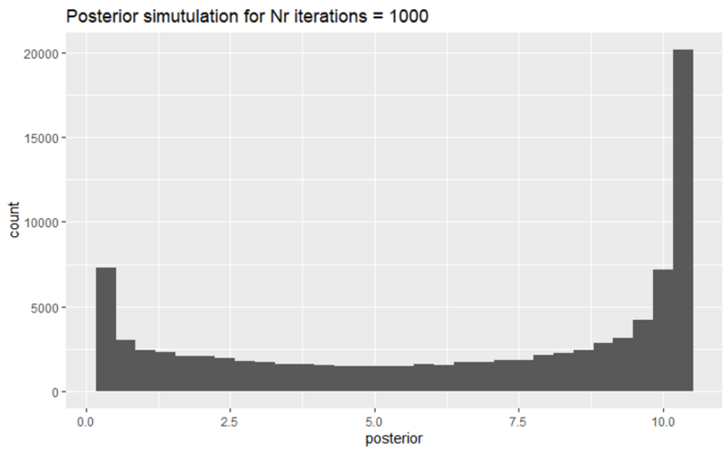
Confidence interval of 95/100



Posterior analysis



Posterior analysis



References

- G. Furioli, A. Pulvirenti, E. Terraneo, G.T. , *Fokker–Planck equations in the modelling of socio-economic phenomena*, Math. Mod. Meth. Appl. Scie. **27** (1) (2017) 115–158 .
- L. Pareschi, G.T. , *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*, Oxford University Press, Oxford (2014) .
- A.K. Gupta, Models of wealth distributions: a perspective. In *Econophysics and sociophysics: trends and perspectives* B.K. Chakrabarti, A. Chakraborti, A. Chatterjee (Eds.) Wiley VHC, Weinheim (2006) 161–190 .
- G.T. , *Kinetic models of opinion formation*, Commun. Math. Sci. **4** (3) (2006) 481–496.

G. Ajmone Marsan, N. Bellomo, L. Gibelli, Stochastic evolutionary differential games toward a systems theory of behavioral social dynamics *Math. Mod. Meth. Appl. Scie.* **26**, (6) (2016) 1051–1093.

A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations, *Commun. Partial Diff. Equa.* **26** (2001), 43–100.

J. Angle, The surplus theory of social stratification and the size distribution of personal wealth. *Social Forces* **65** (1986) 293–326.

J. Angle, The inequality process as a wealth maximizing process. *Physica A* **367** (2006) 388–414.

F. Bassetti and G. Toscani.
Explicit equilibria in a kinetic model of gambling.
Phys. Rev. E **81** (2010) 066115.

F. Bassetti and G. Toscani.

Explicit equilibria in bilinear kinetic models for socio-economic interactions.
ESAIM: Proc. and Surveys **47** (2014) 1–16.

F. Bassetti and G. Toscani.
Mean field dynamics of collisional processes with duplication, loss and copy, *Math. Mod. Meth. Appl. Scie.* **25** (10) (2015) 1887–1925.

E. Battistin, R. Blundell and A. Lewbel, Why is consumption more log normal than income? Gibrat's law revisited, *Journal of Political Economy* **117** (6) (2009) 1140–1154.

N. Bellomo, D. Knopoff and J. Soler.
On the difficult interplay between life, "complexity", and mathematical sciences.
Math. Models Methods Appl. Sci. **23** (2013) 1861–1913.

N. Bellomo, N.K. Li and P.K. Maini.
On the foundations of cancer modelling: selected topics, speculations, and perspectives.

Math. Models Methods Appl. Sci. **18** (2008) 593–646.

J. Brainard and D.E. Burmaster, Bivariate distributions for height and weight of men and women in the United States, *Risk Analysis* **12** (1992) 267–275.

D.E. Burmaster and E.A. Crouch, Lognormal distributions for body weight as a function of age for males and females in the United States, 1976–1980, *Risk Anal.* **17** (4) (1997) 499–505.

N. Bellomo and J. Soler.
On the mathematical theory of the dynamics of swarms viewed as complex systems.
Math. Models Methods Appl. Sci. Suppl. **22** (2012) 1140006 (29 pages).

A. Bellouquid, E. De Angelis and D. Knopoff.
From the modelling of the immune hallmarks of cancer to a black swan in biology.
Math. Models Methods Appl. Sci. **23** (2013) 949–978.

- E. Ben-Naim, P. L. Krapivski and S. Redner, Bifurcations and patterns in compromise processes. *Physica D* **183**, (2003) 190–204.
- E. Ben-Naim, P. L. Krapivski, R. Vazquez and S. Redner, Unity and discord in opinion dynamics. *Physica A* **330**, (2003) 99–106.
- E. Ben-Naim, Opinion dynamics: rise and fall of political parties. *Europhys. Lett.* **69**, (2005) 671–677.
- M.L. Bertotti and M. Delitala, On a discrete generalized kinetic approach for modelling persuader's influence in opinion formation processes. *Math. Comp. Model.* **48**, (2008) 1107–1121.
- M. Bisi, G. Spiga and G. Toscani, Kinetic models of conservative economies with wealth redistribution. *Commun. Math. Sci.* **7** (4) 901–916 (2009)
- G. Bobkov and M. Ledoux, Weighted Poincaré-type Inequalities for Cauchy and other convex measures. *Ann. Probability* **37** (2) (2009) 403–427

J.F. Bouchaud and M. Mezard, Wealth condensation in a simple model of economy. *Physica A*, **282** (2000) 536–545.

L. Boudin and F. Salvarani, The quasi-invariant limit for a kinetic model of sociological collective behavior. *Kinetic Rel. Mod.* **2**, (2009) 433–449.

L. Boudin and F. Salvarani, A kinetic approach to the study of opinion formation. *ESAIM: Math. Mod. Num. Anal.* **43**, (2009) 507–522.

L. Boudin, A. Mercier and F. Salvarani, Conciliatory and contradictory dynamics in opinion formation. *Physica A* **391**, (2012) . 5672–5684.

C. Brugna and G. Toscani, Kinetic models of opinion formation in the presence of personal conviction, *Phys. Rev. E* **92**, (2015) 052818.

M. Burger, L. Caffarelli, P.A. Markowich and M-T. Wolfram.
On a Boltzmann-type price formation model.
Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **469** (2013).

M. Burger, L. Caffarelli, P.A. Markowich and M-T. Wolfram.

On the asymptotic behavior of a Boltzmann-type price formation model.
Commun. Math. Sci. **12** (2014) 1353–1361.

M.J. Cáceres and G. Toscani.
Kinetic approach to long time behavior of linearized fast diffusion equations.
J. Stat. Phys. **128** (2007) 883–925.

J.A. Carrillo, M. Fornasier, J. Rosado and G. Toscani.
Asymptotic flocking dynamics for the kinetic Cucker-Smale model.
SIAM J. Math. Anal. **42** (2010) 218–236.

J.A. Carrillo, M. Fornasier, G. Toscani and F. Vecil.
Particle, kinetic, and hydrodynamic models of swarming.
In *Mathematical modelling of collective behavior in socio-economic and life sciences*, Model. Simul. Sci. Eng. Technol., pages 297–336. Birkhäuser Boston, Inc., Boston, MA, 2010.

C. Castellano, S. Fortunato and V. Loreto, Statistical physics of social dynamics. *Rev. Mod. Phys.* **81** (2009) 591–646.

A. Chakraborti, Distributions of money in models of market economy. *Int. J. Modern Phys. C* **13**, (2002) 1315–1321.

A. Chakraborti and B.K. Chakrabarti, Statistical Mechanics of Money: Effects of Saving Propensity. *Eur. Phys. J. B* **17**, (2000) 167–170.

A. Chatterjee, B.K. Chakrabarti and S.S. Manna, Pareto law in a kinetic model of market with random saving propensity. *Physica A* **335** (2004), 155–163.

A. Chatterjee, B.K. Chakrabarti and R.B. Stinchcombe, Master equation for a kinetic model of trading market and its analytic solution. *Phys. Rev. E* **72**, (2005) 026126.

H. Chernoff, A note on an inequality involving the normal distribution. *Ann. Probab.* **9** (3) (1981) 533–535.

- V. Comincioli, L. Della Croce and G. Toscani, A Boltzmann-like equation for choice formation. *Kinetic Rel. Mod.* **2**, (2009) 135–149.
- S. Cordier, L. Pareschi and C. Piatecki, Mesoscopic modelling of financial markets. *J. Stat. Phys.* **134** (1), (2009) 161–184.
- S. Cordier, L. Pareschi and G. Toscani, On a kinetic model for a simple market economy. *J. Stat. Phys.* **120**, (2005) 253-277.
- I. Csiszár, Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis von Markoffschen Ketten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8**, (1963) 85–108.
- F. Cucker and E. Mordecki,
  Flocking in noisy environments.
  J. Math. Pures Appl. **89** (2008) 278–296.
- F. Cucker and S. Smale,
  Emergent behavior in flocks.
  IEEE Trans. Automat. Control **52** (2007) 852–862.

- F. Cucker and S. Smale.
  On the mathematics of emergence.
  Jpn. J. Math. **2** (2007) 197–227.

- G. Deffuant, F. Amblard, G. Weisbuch and T. Faure, How can extremism prevail? A study based on the relative agreement interaction model. *J. Art. Soc. Soc. Sim.* **5**, (2002) 1.

- A. Dragulescu and V.M. Yakovenko, Statistical mechanics of money. *Eur. Phys. Jour. B* **17**, (2000) 723-729.

- B. Düring, D. Matthes and G. Toscani, Kinetic Equations modelling Wealth Redistribution: A comparison of Approaches. *Phys. Rev. E*, **78**, (2008) 056103.

- B. Düring, D. Matthes and G. Toscani, A Boltzmann-type approach to the formation of wealth distribution curves. (Notes of the Porto Ercole School, June 2008) *Riv. Mat. Univ. Parma* (1) **8** (2009) 199–261.

B. Düring and G. Toscani, International and domestic trading and wealth distribution. *Commun. Math. Sci.*, **6**/4: (2008) 1043–1058.

B. Düring, P.A. Markowich, J-F. Pietschmann and M-T. Wolfram. Boltzmann and Fokker-Planck equations modelling opinion formation in the presence of strong leaders. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **465** (2009) 3687–3708.

U. Eco, Memoria e dimenticanza. (2011), in http://www.3dnews.it/node/1824.

J. Eeckhout, Gibrat's law for (all) cities, *American Economic Review* **94** (2004) 1429–1451.

W. Feller, Two singular diffusion problems. *Ann. Math.* **54** (2) (1951) 173–182.

W. Feller, The Parabolic Differential Equations and the Associated Semi-Groups of Transformations, *Ann. Math.* **55** (3) (1952) 468–519

- W. Feller,
  An introduction to probability theory and its applications. Vol. I.
  (John Wiley & Sons Inc. 1968).

- G. Furioli, A. Pulvirenti, E. Terraneo and G. Toscani.
  The grazing collision limit of the inelastic Kac model around a Lévy-type equilibrium.
  SIAM J. Math. Anal. **44** (2012) 827–850.

- E. Gabetta and E. Regazzini,
  About the gene families size distribution in a recent model of genome evolution.
  Math. Models Methods Appl. Sci. **20** (2010) 1005–1020.

- S. Galam, Y. Gefen and Y. Shapir, Sociophysics: A new approach of sociological collective behavior. I. Mean-behaviour description of a strike *J. Math. Sociology* **9**, (1982) 1–13.

S. Galam and S. Moscovici, Towards a theory of collective phenomena: consensus and attitude changes in groups. *Euro. J. Social Psychology* **21**, (1991) 49–74.

S. Galam, Rational group decision making: A random field Ising model at $T = 0$. *Physica A* **238**, (1997) 66–80.

S. Galam and J.D. Zucker, From individual choice to group decision-making. *Physica A* **287**, (2000) 644–659.

U. Garibaldi, E. Scalas and P. Viarengo, Statistical equilibrium in simple exchange games II. The redistribution game. *Eur. Phys. Jour. B* **60**(2) (2007) 241–246.

J.C. Gonzalez–Avella, V.M. Eguiluz, M. Marsili, F. Vega–Redondo and M. San Miguel, Threshold learning dynamics in social networks. *PLoS ONE* **6** (5) (2011) e20207.

J. Grilli, M. Romano, F. Bassetti and M. Cosentino Lagomarsino,

Cross-species gene-family fluctuations reveal the dynamics of horizontal transfers.
Nucleic Acids Research **42** (2014) 6850–60.

A.K. Gupta, Models of wealth distributions: a perspective. In *Econophysics and sociophysics: trends and perspectives* B.K. Chakrabarti, A. Chakraborti, A. Chatterjee (Eds.) Wiley VHC, Weinheim (2006) 161–190.

S-Y. Ha, E. Jeong, J-H. Kang and K. Kang,
Emergence of multi-cluster configurations from attractive and repulsive interactions.
Math. Models Methods Appl. Sci. **22** (2012) 1250013 (42 pages).

S-Y. Ha, M-J. Kang and B. Kwon,
A hydrodynamic model for the interaction of Cucker-Smale particles and incompressible fluid.
Math. Models Methods Appl. Sci. **24** (2014) 2311–2359.

S-Y. Ha and E. Tadmor,

From particle to kinetic and hydrodynamic descriptions of flocking. *Kinet. Relat. Models* **1** (2008) 415–435.

- B. Hayes, Follow the money. *American Scientist* **90** (5), (2002) 400–405.
- P. Herd, K. Holden and Y.T. Su, The links between early-life cognition and schooling and late-life financial knowledge. *The Journal of Consumer Affairs*, Fall 2012, (2012) 411–435
- M. A. Huynen and E. van Nimwegen, The frequency distribution of gene family size in complete genomes. *Mol. Biol. Evol.* **15** (1998) 583–589.
- I. A. Ibragimov and M.. Rradavichyus, On large deviation probabilities for maximum likelihood estimators. (Russian) *Dokl. Akad. Nauk SSSR* **5** (1981), 1048–1052
- S. Ispolatov, P.L. Krapivsky and S. Redner, Wealth distributions in asset exchange models. *Eur. Phys. Jour. B* **2**, (1998) 267–276.

O. Johnson and A. Barron, Fisher information inequalities and the central limit theorem. *Probab. Theory Related Fields* **129** (3) (2004) 391–409.

X. Jin, Y. Zhang, F. Wang, L. Li, D. Yao, Y. Su and Z. Wei, Departure headways at signalized intersections: A log-normal distribution model approach, *Transportation Research Part C* **17** (2009) 318–327.

D. Kahneman and A. Tversky. Prospect Theory: An Analysis of Decision under Risk, *Econometrica*, **47** (2) (1979) 263–292.

G. P. Karev, Y. I. Wolf, A. Y. Rzhetsky, F. S. Berezovskaya and E. V. Koonin,
Birth and death of protein domains: A simple model of evolution explains power law behaviour.
BMC Evol. Biol. **2** (2002) 1–26.

E. Kashdan and L. Pareschi,
Mean field mutation dynamics and the continuous Luria-Delbrück distribution.

Math. Biosci. **240** (2012) 223–230.

C. A. Klaassen, On an Inequality of Chernoff. *Ann. Probability* **13** (3) (1985) 966–974.

S. Kullback,
Information Theory and Statistics.
(John Wiley, 1959).

J. Lin, Divergence measures based on the Shannon entropy. *IEEE Trans. Inf. Theo.* **37**, (1991) 145–151.

D.E. Lea and C.A. Coulson.
The distribution of the numbers of mutants in bacterial populations.
Journal of genetics **49** (1949) 264–285.

C. Le Bris and P.L. Lions Existence and Uniqueness of Solutions to Fokker–Planck Type Equations with Irregular Coefficients, *Communications in Partial Differential Equations* **33** (7) (2008) 1272–1317.

C.F. Lo, Dynamics of Fokker–Planck equation with logarithmic coefficients and its application in econophysics, *Chin. Phys. Lett.* **27** (8) (2010) 080503.

S.E. Luria and M. Delbrück,
Interference between inactivated bacterial virus and active virus of the same strain and of a different strain.
Arch. Biochem (1942) 207–218.

S.E. Luria and M. Delbrück,
Mutations of bacteria from virus sensitivity to virus resistance.
Genetics (1943) 491-511.

D. Maldarella and L. Pareschi, Kinetic models for socio–economic dynamics of speculative markets, *Physica A*, **391** (2012) 715–730 .

A.C.R. Martins and S. Galam, Building up of individual inflexibility in opinion dynamics. *Phys. Rev. E*, **87** (2013) 042807.

- D. Matthes, A. Juengel and G.Toscani, Convex Sobolev inequalities derived from entropy dissipation. *Arch. Rat. Mech. Anal.* **199** (2) (2011) 563–596.
- D. Matthes and G. Toscani,
  On steady distributions of kinetic models of conservative economies.
  J. Stat. Phys. **130** (2008) 1087–1117.
- G. Naldi, L. Pareschi and G. Toscani,
  Mathematical modeling of collective behavior in socio-economic and life sciences.
  (Springer Verlag, Heidelberg, 2010).
- F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, *J. Funct. Anal.* **173** (2) (2000) 361–400.
- L. Pareschi, G. Toscani, Wealth distribution and collective knowledge. A Boltzmann approach, *Phil. Trans. R. Soc. A* **372**, 20130396, 6 October (2014).

- L. Pareschi and G. Toscani,
  Interacting multiagent systems. Kinetic equations & Monte Carlo methods.
  (Oxford University Press, Oxford, 2013).
- V. Pareto, *Cours d'Économie Politique*. Lausanne and Paris (1897).
- K. Pesz, A class of Fokker-Planck equations with logarithmic factors in diffusion and drift terms, *Journal of Physics A* **35** (8) (2002) 1827–1832.
- K. Portier, J.K. Tolson and S.M. Roberts, Body weight distributions for risk assessment, *Risk Analysis* **27** (1) (2007) 11–26.
- F.W. Preston, Pseudo-lognormal distributions, *Ecology* **62** (2) (1981) 355–364.
- A. Pulvirenti, G. Toscani, Asymptotic properties of the inelastic Kac model, *J. Statist. Phys.*, **114** (2004) 1453–1480
- R. Rudnicki and J. Tiuryn,

Size distribution of the gene families in a genome. *Math. Models Methods Appl. Sci.* **24** (2014) 697–717.

- P. Sánchez Moreno, A. Zarzo and J. S. Dehesa, Jensen divergence based on Fisher's information. *J. Phys. A: Math. Theor.* **45** (12) (2012) 125305
- E. Scalas, U. Garibaldi and S. Donadio, Statistical equilibrium in the simple exchange games I. Methods of solution and application to the Bennati–Dragulescu–Yakovenko (BDY) game. *Eur. Phys. J. B* **53** (2006) 267–272.
- F. Slanina, Inelastically scattering particles and wealth distribution in an open economy. *Phys. Rev. E* **69**, (2004) 046102.
- K. Sznajd–Weron and J. Sznajd, Opinion evolution in closed community. *Int. J. Mod. Phys. C* **11**, (2000) 1157–1165.
- R.C. Teevan and R.C. Birney, *Readings for Introductory Psychology* Harcourt, Brace & World, New York, (1965).

F. Topsøe,
Some Inequalities for Information Divergence and Related Measures of Discrimination.
IEEE Trans. Inf. Theory **46** (4) (2000) 1602–1609.

M. Torregrossa and G. Toscani, On a Fokker–Planck equation for wealth distribution, (preprint) (2016)

G. Toscani, Entropy dissipation and the rate of convergence to equilibrium for the Fokker-Planck equation, *Quart. Appl. Math.*, **LVII** (1999), 521–541

G. Toscani,
Kinetic models of opinion formation.
Commun. Math. Sci. **4** (2006) 481–496.

G. Toscani, An information-theoretic proof of Nash's inequality, *Rend. Lincei Mat. Appl.*, **24** (2013) 83–93

G. Toscani, C. Brugna and S. Demichelis, Kinetic models for the trading of goods. *J. Stat. Phys.* **151** 549–566 (2013).

G. Toscani,
A kinetic description of mutation processes in bacteria.
Kinet. Relat. Models **6** (2013) 1043–1055.

G. Toscani, Kinetic and mean field description of Gibrat's law, *Physica A*, **461** (2016) 802–811

C. Villani, *Contribution à l'étude mathématique des équations de Boltzmann et de Landau en théorie cinétique des gaz et des plasmas*. PhD Thesis, Univ. Paris-Dauphine (1998).

C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal*, **143** (1998) 273–307.

C. Villani, A Review of Mathematical Topics in Collisional Kinetic Theory, in S. Friedlander and D. Serre (eds), *Handbook of Mathematical Fluid Dynamics*, Vol. 1, New York: Elsevier (2002)

V. M. Zolotarev, Properties and relations of certain types of metrics. (Russian. English summary) Studies in mathematical statistics, 3. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **87** (1979), 18–35, 206, 212.